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ON THE NON-HOMOGENEOUS BI-QUADRATIC EQUATION WITH FOUR UNKNOWNS $8XY + 5Z^2=5W^4$

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ABSTRACT

This paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous bi-quadratic equation with four unknowns given by $8xy+5z^2=5w^4$.*We obtain infinitely many non-zero integer solutions of the equation by introducing the linear transformations* $x=u+v, y=u-v, z=v$.

KEYWORDS: *Bi-Quadratic Equation with Four Unknowns, Integral Solutions, Non Homogeneous bi-Quadratic, Linear Transformations*

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INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, bi-quadratic Diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-5]. In this context, one may refer [6-18] for various problems on the bi-quadratic Diophantine equations with four variables. However, often we come across non-homogeneous bi-quadratic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous equation with four unknowns given by $8xy+5z^2=5w^4$.

Method of Analysis

Consider the bi-quadratic equations with four unknowns

$$8xy+5z^2=5w^4 \tag{1}$$

The process of obtaining different choices of non-zero distinct integer solutions to (1) is illustrated below:

Choice 1:

Introducing the linear transformations

$$x = u + v, y = u - v, z = v \tag{2}$$

in (1), it is written as

$$8u^2 - 3v^2 = 5w^4 \tag{3}$$

Again, employing the linear transformations

$$u = X + 3T \; , \; v = X + 8T \; , \tag{4}$$

(3) is written as

$$X^2 - 24T^2 = w^4 \tag{5}$$

Express (5) as the system of double equations as shown in Table 1 below:

Table 1: System of Double Equations

System	I	II	III	IV
$X + w^2$	$6T^2$	$8T$	$6T$	$12T$
$X - w^2$	4	$3T$	$4T$	$2T$

Consider System I.

Solving for X and w , we get

$$X = 3T^2 + 2 \tag{6}$$

$$w^2 = 3T^2 - 2 \tag{7}$$

Now, observe that (7) is a negative Pellian equation whose solutions are obtained as illustrated below:

The smallest positive integer solution to (7) is

$$T_0 = 1 \; , \; w_0 = 1 \tag{8}$$

To obtain the other solutions to (7), consider the corresponding Pellian equation

$$w^2 = 3T^2 + 1 \tag{9}$$

The smallest positive integer solution to (9) is

$$\tilde{T}_0 = 1 \; , \; \tilde{w}_0 = 2$$

Whose, general solution is given by

$$\tilde{w}_n = \frac{1}{2} f_n \; , \; \tilde{T}_n = \frac{1}{2\sqrt{3}} g_n$$

Where, $f_n = \left(2 + \sqrt{3}\right)^{n+1} + \left(2 - \sqrt{3}\right)^{n+1}$

$$g_n = \left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}, \ n = 0,1,2.....$$

Applying Brahmagupta lemma between the solutions $\left(T_0, w_0\right)$ and $\left(\tilde{T}_n, \tilde{w}_n\right)$, the other integer solutions of (7) are given by

$$T_{n+1} = \frac{1}{2}f_n + \frac{1}{2\sqrt{3}}g_n \qquad (10)$$

$$w_{n+1} = \frac{1}{2}f_n + \frac{\sqrt{3}}{2}g_n \qquad (11)$$

Using (10) in (6), we have

$$X_{n+1} = 3\left(\frac{1}{2}f_n + \frac{1}{2\sqrt{3}}g_n\right)^2 + 2$$

In view of (4) and (2), the values of x, y and z are represented by

$$\left.\begin{aligned} x_{n+1} &= 6\left(\frac{1}{2}f_n + \frac{1}{2\sqrt{3}}g_n\right)^2 + 11\left(\frac{1}{2}f_n + \frac{1}{2\sqrt{3}}g_n\right) + 4 \\ y_{n+1} &= -5\left(\frac{1}{2}f_n + \frac{1}{2\sqrt{3}}g_n\right) \\ z_{n+1} &= 3\left(\frac{1}{2}f_n + \frac{1}{2\sqrt{3}}g_n\right)^2 + 8\left(\frac{1}{2}f_n + \frac{1}{2\sqrt{3}}g_n\right) + 2 \end{aligned}\right\} \qquad (12)$$

Thus, (12) and (11) represent the integer solutions to (1)

Consider System II

Solving for X and w, we get $X = \dfrac{11T}{2}$, $w^2 = \dfrac{5T}{2}$

Taking

$$T = 10k^2$$

we have

$$X = 55k^2$$

$$w = 5k \qquad (13)$$

In view of (4) and (2), the values of x , y and z are represented by

$$x = 220k^2 \ , \ y = -50k^2, \ z = 135k^2 \qquad (14)$$

Thus, (14) and (13) represent the integer solutions to (1)

Consider System III

Solving for X and w, we get $X = 5T$, $w^2 = T$

Taking

$T = k^2$

we have

$X = 5k^2$

$w = k$ (15)

In view of (4) and (2), the values of x , y and z are represented by

$x = 21k^2$, $y = -5k^2$, $z = 13k^2$ (16)

Thus, (16) and (15) represent the integer solutions to (1).

Consider System IV. Solving for X and w, we get $X = 7T$, $w^2 = 5T$

Taking

$T = 5k^2$

We have

$X = 35k^2$

$w = 5k$ (17)

In view of (4) and (2), the values of x , y and z are represented by

$x = 125k^2$, $y = -25k^2$, $z = 75k^2$ (18)

Thus, (17) and (18) represent the integer solutions to (1).

Choice 2:

Introducing the linear transformation

$z = x - y$ (19)

in (1), it becomes

$5x^2 - 2xy + (5y^2 - 5w^4) = 0$ (20)

Considering (20) as a quadratic expression in x and solving for x, one obtains

$x = \frac{1}{5}(y \pm \quad)$ (21)

Where

$$^{2} = 25w^4 - 24y^2 \qquad (22)$$

Now, (22) is written in the form of ratio as

$$\frac{+w^2}{w^2 + y} = \frac{24(w^2 - y)}{-w^2} = \frac{M}{N},\ N > 0$$

which is equivalent to the system of double equations

$$N \quad + (N - M)w^2 - My = 0 \qquad (23)$$

$$-M \quad + (24N + M)w^2 - 24Ny = 0 \qquad (24)$$

Applying the method of cross- multiplication between (23) and (24) we get

$$= M^2 - 24N^2 + 48MN \qquad (25)$$

$$w^2 = M^2 + 24N^2 \qquad (26)$$

$$y = -M^2 + 24N^2 + 2MN \qquad (27)$$

Substituting (25), (27) in (21) and taking the positive sign, we get

$$x = 10MN \qquad (28)$$

and from (19),

$$z = M^2 - 24N^2 + 8MN \qquad (29)$$

Now, observe that (26) is satisfied by

$$M = 24s^2 - t^2,\ N = 2st \qquad (30)$$

$w = 24s^2 + t^2$ where s, t are integers. (31)

Substituting (30) in (27), (28) & (29), we get

$$\left.\begin{aligned} x &= 20st(24s^2 - t^2) \\ y &= -(24s^2 - t^2)^2 + 96s^2t^2 + 4st(24s^2 - t^2) \\ z &= (24s^2 - t^2)^2 - 96s^2t^2 + 16st(24s^2 - t^2) \end{aligned}\right\} \qquad (32)$$

Thus, (31) and (32) give integer solutions of (1).

Note:

It is to be noted that (22) is also expressed in the form of ratios as follows:

i. $\frac{+w^2}{24(w^2+y)} = \frac{w^2-y}{-w^2} = \frac{M}{N},\ N>0$

ii. $\frac{+w^2}{4(w^2+y)} = \frac{6(w^2-y)}{-w^2} = \frac{M}{N},\ N>0$

iii. $\frac{+w^2}{6(w^2+y)} = \frac{4(w^2-y)}{-w^2} = \frac{M}{N},\ N>0$

iv. $\frac{+w^2}{3(w^2+y)} = \frac{8(w^2-y)}{-w^2} = \frac{M}{N},\ N>0$

v. $\frac{+w^2}{8(w^2+y)} = \frac{3(w^2-y)}{-w^2} = \frac{M}{N},\ N>0$

vi. $\frac{+w^2}{2(w^2+y)} = \frac{12(w^2-y)}{-w^2} = \frac{M}{N},\ N>0$

vii. $\frac{+w^2}{12(w^2+y)} = \frac{2(w^2-y)}{-w^2} = \frac{M}{N},\ N>0$

Solving each of the above ratios as presented above, one obtains different sets of integer solutions to (1).

However, there are other sets of solutions to (26) leading to other choices of solutions to (1) that are illustrated below:

We can write (26) as the system of double equations as shown in following Table 2:

Table 2: System of Double Equations

System	I	II	III	IV	V	VI
$w+M$	N^2	$12N^2$	$6N^2$	$8N$	$6N$	$12N$
$w-M$	24	2	4	$3N$	$4N$	$2N$

For simplicity and brevity, we present below the integer solutions to (1) obtained on solving each of the above system of equations.

System I:

$$x = 40p(p^2-6)$$
$$y = -4(p^2-6)^2 + 96p^2 + 8p(p^2-6)$$
$$z = 4(p^2-6)^2 - 96p^2 + 32p(p^2-6)$$
$$w = 2p^2 + 12$$

System II:

$$x = 60N^3 - 10N$$
$$y = -(6N^2 - 1)^2 + 24N^2 + 2N(6N^2 - 1)$$
$$z = (6N^2 - 1)^2 - 24N^2 + 8N(6N^2 - 1)$$
$$w = 6N^2 + 1$$

System III:

$$x = 10N(3N^2 - 2)$$
$$y = -(3N^2 - 2)^2 + 24N^2 + 2N(3N^2 - 2)$$
$$z = (3N^2 - 2)^2 - 24N^2 + 8N(3N^2 - 2)$$
$$w = 3N^2 + 2$$

System IV:

$$x = 100k^2\ ,\ \ y = 91k^2,\ \ z = 9k^2,\ \ w = 11k$$

System V:

$$x = 10N^2,\ \ y = 25N^2,\ \ z = 15N^2,\ \ w = 5N$$

System VI:

$$x = 50N^2,\ \ y = 9N^2,\ z = 41N^2,\ \ w = 7N$$

Choice 3:

Substituting

$$x = 5 * 2^{2\sim -1}\, y,\ \ (\sim > 1) \tag{33}$$

in (1),

It is written as

$$\left(2^{2\sim -1}\, y\right)^2 + z^2 = w^4 \tag{34}$$

Note that, the above equation is similar to the well-known Pythagorean equation. Employing the most cited solutions of the Pythagorean equation, one obtains the following two sets of solution to (1):

Set 1

$$x = 5 * 2^{3\sim}\, uv(u^2 - v^2)$$
$$y = 2uv * 2^{\sim}(u^2 - v^2)$$
$$z = 2^{2\sim}\left[4u^2v^2 - (u^2 - v^2)^2\right]$$
$$w = 2^{\sim}(u^2 + v^2)$$

Set 2

$$x = 5 * 2^{3\sim}\left(6p^2q^2 - p^4 - q^4\right)$$
$$y = 2^{\sim+1}\left(6p^2q^2 - p^4 - q^4\right)$$
$$z = 16 * 2^{2\sim} * pq\left(p^2 - q^2\right)$$
$$w = 2^{\sim+1}\left(p^2 + q^2\right)$$

When $\sim = 1$ in (30), the corresponding solutions to (1) are obtained as below:

Set 3:

$$x = 10uv\left(u^2 - v^2\right)$$
$$y = uv\left(u^2 - v^2\right)$$
$$z = 4u^2v^2 - \left(u^2 - v^2\right)^2$$
$$w = u^2 + v^2$$

Set 4:

$$x = 10\left(6p^2q^2 - p^4 - q^4\right)$$
$$y = \left(6p^2q^2 - p^4 - q^4\right)$$
$$z = 16pq\left(p^2 - q^2\right)$$
$$w = 2\left(p^2 - q^2\right)$$

CONCLUSION

In this paper, an attempt has been made to obtain many non-zero distinct integer solutions to the non-homogeneous bi-quadratic equation with four unknowns represented by $8xy + 5z^2 = 5w^4$. As bi-quadratic equations are rich in variety, the researchers may search for integer solutions to the other types of bi-quadratic equations with variables greater than or equal to four.

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